Zero modes in non local domain walls

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Abstract

We generalize the Callan-Harvey mechanism to the case of actions with a non local mass term for the fermions. Using a $2+1$-dimensional model as a concrete example, we show that both the existence and properties of localized zero modes can also be consistently studied when the mass is non local. We derive some general properties from a study of the resulting integral equations, and consider their realization in a concrete example.

The Callan-Harvey mechanism [1] explains the existence and properties of fermionic zero modes that appear whenever the mass of a Dirac field in $2k+1$ dimensions ($k = 1, 2, \cdots$) has a domain-wall like defect. The zero modes due to this phenomenon are localized, concentrated around the domain wall, and chiral from the point of view of the domain-wall world-volume (a $2k$-dimensional theory).

This mechanism has found many interesting applications, both to phenomenological issues [2] and theoretical elaborations [3, 4]. Dynamical domain walls have been considered in [5], and the case of a (dynamical) supersymmetric model with this kind of configuration has been discussed in [6]. Remarkably, lattice versions of the zero modes, the so-called 'domain wall fermions' [7] have also been precursors of the overlap Dirac operator [8], a sensible way to put chiral fermions on a spacetime lattice.

In spite of the fact that the Callan-Harvey mechanism has been extended in many directions, the existence of chiral zero modes has been so far only studied for the case of *local* mass terms, namely, those where the mass is a local function of the spacetime coordinates. In this article, we relax that condition and consider the case of a non local mass in $2+1$ dimensions. This is a non trivial modification which does, however, arise naturally in some applications. For example, when the fermionic action is dressed by quantum effects, the corrections usually adopt the form of non local form-factors.

Let us first review the standard Callan-Harvey mechanism [1] when one considers a fermion field in 2+1 dimensions coupled to a domain wall like defect, with the (Euclidean) action

$$S(\bar{\psi}, \psi) \; = \; \int d^3x \, \bar{\psi} \, [\not{\partial} \, + \, m(x)] \, \psi \; . \tag{1}$$

We use Euclidean coordinates $x = (x_0, x_1, x_2)$ (x_0 is the Euclidean time) and $\not{\partial} = \gamma_\mu \partial_\mu$, where the γ-matrices are chosen according to the convention:

$$\gamma_0 \; = \; \sigma_3 \quad \gamma_1 \; = \; \sigma_1 \quad \gamma_2 \; = \; \sigma_2 \; . \tag{2}$$

The *local* mass $m(x)$ contains a topological defect; in the simplest case of a rectilinear static defect [3], they have the characteristic shape:

$$m(x) \sim \Lambda \, \sigma(x_2) \; , \tag{3}$$

where $\sigma(x_2) \equiv \mathrm{sign(x_2)}$. Therefore the domain wall, which is the interface between two regions with different signs for $m(x)$, is the x_1 axis.

From the Dirac operator

$$\mathcal{D} \ = \not{\partial}_x + m(x_2) \ \ , \tag{4}$$

we can construct the hermitian operator $\mathcal{H} = \mathcal{D}^\dagger \, \mathcal{D}$. The form of $\mathcal{H}$ suggests the introduction of the adjoint operators

$$a \ = \ \partial_2 + m(x_2) \ \ , \quad a^\dagger \ = \ -\partial_2 + m(x_2) \tag{5}$$

in terms of which

$$\mathcal{H} \ = \ (a^\dagger a - \hat{\not{\partial}}^2) P_L + (a a^\dagger - \hat{\not{\partial}}^2) P_R \tag{6}$$

and

$$\mathcal{D} \ = \ (a + \hat{\not{\partial}}) P_L + (a^\dagger + \hat{\not{\partial}}) P_R \tag{7}$$

where $P_L = \frac{1}{2}(1+\gamma_2)$, $P_R = \frac{1}{2}(1-\gamma_2)$. Expanding $\psi(x)$ in the complete set of eigenfunctions of $a^\dagger a$ and $a a^\dagger$, there appears [3] a massless left fermion, localized over the domain wall. Its x_2 dependence is dictated by the fact that it is a zero mode of the a operator, and it dominates the low energy dynamics of the system.

We want to generalize this phenomenon to include a non local mass. Namely, rather than (4), the Dirac operator shall be

$$\tilde{\mathcal{D}}(x,y) \ = \not{\partial}_x \, \delta(x-y) + M(x,y) \ \ . \tag{8}$$

Little can be said about the existence of a fermionic zero mode before we make some hypotheses to restrict the form of $M(x,y)$. We assume that the system has translation invariance in the coordinates $\hat{x} \equiv (x_0, x_1)$ and that $M(x,y)$ consists of a local domain wall like part, plus a non local term with a strength controlled by a parameter λ:

$$M(x,y) \ = \ m(x_2) \, \delta(x-y) - \lambda \int \frac{d^2\hat{k}}{(2\pi)^2} \, e^{i\hat{k}\cdot(\hat{x}-\hat{y})} \gamma_k(x_2,y_2) \ \ . \tag{9}$$

We are looking for a zero mode $\Psi(x)$, so that:

$$\langle x|\tilde{\mathcal{D}}|\Psi\rangle \ = \ \int \, d^3y \, \tilde{\mathcal{D}}(x,y)\Psi(y) = 0 \ \ . \tag{10}$$

Taking advantage of the translation invariance in $\hat{x}$, we use 'separation of variables' to look for solutions of the form:

$$\Psi(x) \ = \ \chi(\hat{x}) \, \psi(x_2) \ \ , \tag{11}$$

where $\chi(\hat{x})$ is a massless spinor, which is left-handed from the point of view of the two-dimensional world defined by $\hat{x}$, i.e.,

$$\hat{\not{\partial}}\chi(\hat{x}) \;=\; 0 \quad P_R\chi(\hat{x}) \;=\; 0 \;. \tag{12}$$

There is an essential difference regarding the space of solutions to the equations above in Euclidean and Minkowski spacetimes. Indeed, in Euclidean spacetime, it leads to analytic functions of $x_0 + ix_1$, while in the Minkowski case one has 'left-mover' solutions. Keeping this distinction in mind, we continue working with the Euclidean version.

Substituting (11) into (10) and comparing with (7), we arrive to a non local version of the kernel for the annihilation operator

$$a(x_2, y_2) \;=\; [\partial_2 + m(x_2)]\,\delta(x_2 - y_2) - \lambda\,\gamma_k(x_2, y_2) \;\;. \tag{13}$$

The zero mode $\psi(x_2)$, must then satisfy the equation

$$\langle x|a|\psi\rangle \;=\; [\partial_2 + m(x_2)]\,\psi(x_2) - \lambda \int_{-\infty}^{+\infty} dy_2\,\gamma_k(x_2, y_2)\psi(y_2) = 0 \;\;. \tag{14}$$

Following the method of variation of parameters, we use the ansatz

$$\psi(x_2) \;=\; \psi_0(x_2)\varphi(x_2) \tag{15}$$

where ψ_0 is the zero mode for the local part in (14), which satisfies

$$[\partial_2 + m(x_2)]\,\psi_0(x_2) \;=\; 0 \qquad \psi_0(x_2) \;=\; N\,exp[-\int_0^{x_2} ds\,m(s)] \;, \tag{16}$$

and N is a normalization constant.

The function $\varphi(x_2)$, which modulates $\psi_0(x_2)$, must then satisfy the equation

$$\partial_2\varphi(x_2) - \lambda \int_{-\infty}^{+\infty} dy_2\,[\psi_0(x_2)^{-1}\,\gamma_k(x_2, y_2)\,\psi_0(y_2)]\;\varphi(y_2) = 0 \;. \tag{17}$$

By integrating over x_2, the previous equation can be written in a more convenient form as an integral equation

$$\varphi(x_2) - \lambda \int_{-\infty}^{+\infty} dy_2\,\tilde{\gamma}_k(x_2, y_2)\varphi(y_2) = \varphi(0) \tag{18}$$

where the kernel $\tilde{\gamma}_k$ is

$$\tilde{\gamma}_k(x_2, y_2) \ = \ \int_0^{x_2} dz_2 \, \psi_0(z_2)^{-1} \, \gamma_k(z_2, y_2) \, \psi_0(y_2) \ . \tag{19}$$

This is a homogeneous integral equation, which can be conveniently rewritten as an equivalent non-homogeneous set of equations. Indeed, introducing linear operators, (18) can be rewritten as follows:

$$(I - \lambda T)\varphi \ = \ c \tag{20}$$

where

$$(I\varphi)(x_2) \equiv \varphi(x_2) \ , \quad c \equiv \varphi(0) \ , \tag{21}$$

$$(T\varphi)(x_2) \equiv \int dy_2 \, \tilde{\gamma}_k(x_2, y_2) \, \varphi(y_2) \ . \tag{22}$$

We see that the problem can be discussed in terms of (20), which is an inhomogeneous system of the Fredholm type [9]. The condition $c \equiv \varphi(0)$ has to be verified, of course, after solving (20) for arbitrary c.

The reason for this procedure is that, had we used the original homogeneous system, we should have had to introduce non compact operators, and the theory for this kind of operator is much poorer than for the compact case.

Physical restrictions imposed on the non local mass naturally lead us to integral equations of the Fredholm type [9]. The 'Fredholm alternative' [9] states that, if $A = I - \lambda T$, where T is a compact operator on a Hilbert space H, then the following alternative holds: (a) either $A\varphi_0 = 0$ has only the trivial solution, in which case $A\varphi = c$ has a unique solution $\forall c \in H$; or (b) $A\varphi_0 = 0$ has q linearly independent solutions $\varphi_i \in H$. Then $A^\dagger \tilde{\varphi}_0 = 0$ also has q linearly independent solutions $\tilde{\varphi}_i \in H$. In this case $A\varphi = c$ is solvable iff $(c, \tilde{\varphi}_i) = 0 \ \ \forall i = 1, \ldots, q$.

In the case (b), the general non-homogeneous solution is

$$\varphi = \varphi_p + \sum_{i=1}^{q} a_i \varphi_i \tag{23}$$

where φ_p is a particular solution and a_i are arbitrary constants.

On the other hand, when the alternative (a) holds true, this implies that the solution shall be unique when c is replaced by $\varphi(0)$. Solutions corresponding to $c \neq \varphi(0)$ are not solutions of the system: $(I - \lambda T)\varphi = c$, $c \equiv \varphi(0)$,

equivalent to the original homogeneous equation (18), and may, therefore, be safely discarded.

Any true solution of the system will also verify a subsidiary equation, obtained by setting $x_2 = 0$ in (18):

$$\int_{-\infty}^{+\infty} dy_2\, \tilde{\gamma}_k(0, y_2)\varphi(y_2) = 0 \tag{24}$$

for any $\lambda \neq 0$. This equation shall be true whenever the equations $(I - \lambda T)\varphi = c$, $c \equiv \varphi(0)$ are both true (since it is derived from them), and any solution will automatically verify it.

In our case, we want to study the effect of the non local term, the strength of which is controlled by the value of λ. Close enough to the local mass case, λ can be made arbitrarily small, so λ^{-1} is not an eigenvalue of T and consequently $A\varphi = 0$ has only the trivial solution. Therefore, if T is a Fredholm operator, (18) will have a unique solution. The functional space H to which $\varphi(x)$ belongs is restricted by the condition

$$\int dx_2 \, \Big(\psi_0(x_2)\Big)^2 \Big(\varphi(x_2)\Big)^2 < \infty \ , \tag{25}$$

because $\psi(x)$ itself has to be normalizable. This becomes a Hilbert space, and it contains the zero mode of the local operator (φ = constant), when the scalar product used in H is the one defined by (25), namely,

$$(f, g) \ \equiv \ \int dx_2 \, \Big(\psi_0(x_2)\Big)^2 [f(x_2)]^* \, g(x_2) \ . \tag{26}$$

When Fredholm's hypotheses are satisfied for a general kernel $\tilde{\gamma}_k(x_2, y_2)$, it is possible to find a perturbative solution. Indeed, writing

$$\varphi(x_2) \ = \ \varphi(0) + \lambda \int_{-\infty}^{\infty} dy_2\, \tilde{\gamma}_k(x_2, y_2)\, \varphi(y_2) \ , \tag{27}$$

and successively replacing $\varphi(y_2)$ by $\varphi(0) + \lambda \int_{-\infty}^{\infty} dy_2\, \tilde{\gamma}_k(x_2, y_2)\varphi(y_2)$ on the right hand side, we obtain the Neumann series [9]

$$\varphi(0) + \lambda\, K_1(x_2)\, \varphi(0) + \cdots + \lambda^n\, K_n(x_2)\, \varphi(0) + \ldots \tag{28}$$

where

$$K_j(x_2) = \int_{-\infty}^{\infty} dy_2\, \gamma^{(j)}(x_2, y_2) \tag{29}$$

and

$$\gamma^{(1)}(x_2, y_2) \;=\; \tilde{\gamma}_k(x_2, y_2) \;\;,\;\; \gamma^{(2)}(x_2, y_2) \;=\; \int_{-\infty}^{\infty} dz_2 \; \tilde{\gamma}_k(x_2, z_2)\tilde{\gamma}_k(z_2, y_2)$$

$$\ldots \;\; \gamma^{(n+1)}(x_2, y_2) \;=\; \int_{-\infty}^{\infty} dz_2 \; \tilde{\gamma}_k(x_2, z_2)\gamma^{(n)}(z_2, y_2) \; . \tag{30}$$

It can be shown [9] that the Neumann series converges uniformly to the solution $\varphi(x)$ when

$$|\lambda| < \frac{1}{\mu} \; , \tag{31}$$

$$\mu^2 = \int_{-\infty}^{\infty} dx_2 \, dy_2 \; [\tilde{\gamma}_k(x_2, y_2)]^2 \; . \tag{32}$$

So choosing λ to satisfy (31), we are allowed to represent φ by the expansion

$$\varphi(x_2) = \varphi(0) + \lambda \, K_1(x_2) \, \varphi(0) + \cdots + \lambda^n \, K_n(x_2) \, \varphi(0) + \ldots \tag{33}$$

To make the previous analysis more concrete, we consider now an example, based on a specific choice of both the local and non local parts of $M(x, y)$ in (9). A natural generalization of the purely local mass case is to have $m(x_2) = \Lambda \, \sigma(x_2)$ and a $\gamma_k(x_2, y_2)$ which is 'strongly diagonal' and symmetric in (x_2, y_2), i.e.:

$$\gamma_k(x_2, y_2) \;=\; \frac{1}{2} \Big[\sigma(x_2) + \sigma(y_2) \Big] \delta_N(x_2 - y_2) \; , \tag{34}$$

where δ_N is an approximation of Dirac's delta: $\delta_N(x_2 - y_2) \to \delta(x_2 - y_2)$ when $N \to \infty$. Note that $\gamma_k(x_2, y_2) \to \mathrm{sign}(x_2) \, \delta(x_2 - y_2)$ when $N \to \infty$, so that the non local term reduces to a local domain wall mass.

We adopt 'natural' units such that $\Lambda = 1$, and study the particular case $|x_2| \leq L = \Lambda^{-1}$. Regarding δ_N, we use a truncation of the one-dimensional completeness relation:

$$\delta_N(x_2, y_2) \;=\; \sum_{n=0}^{N} \varphi_n(x_2) \, \varphi_n^{\dagger}(y_2) \; , \tag{35}$$

where $\{\varphi_n\}$ is a complete set of functions. We chose $\varphi_n(x)$ to be the harmonic oscillator's eigenfunctions.

The normalized zero mode corresponding to the local part is

$$\psi_0(x_2) \;=\; N_0 \, \exp\Big[- \int_0^{x_2} dt \, m(t) \Big] \;=\; \frac{1}{\sqrt{1-e^{-2}}} \, e^{-|x_2|} \;, \tag{36}$$

and the corrected zero mode $\psi(x_2) \;=\; \psi_0(x_2)\, \varphi(x_2)$ is then determined by $\varphi(x_2)$. The integral equation for $\varphi(x_2)$ becomes

$$\varphi(x_2) \;=\; \varphi(0) + \lambda \int_{-1}^{1} dy_2 \; \tilde{\gamma}(x_2, y_2)\, \varphi(y_2) \tag{37}$$

where

$$\tilde{\gamma}(x_2, y_2) \;=\; \frac{1}{2} \int_0^{x_2} dz_2 \; e^{|z_2|} \Big[\sigma(z_2) \;+\; \sigma(y_2) \Big] \\ \times \Big[\sum_{n=0}^{N} \varphi_n(z_2)\, \varphi_n^\dagger(y_2) \Big] \, e^{-|y_2|} \;. \tag{38}$$

From equations (31) and (32), we see that the Neumann series converges, in this case, for $|\lambda| < 0.0990$. So, we assume that $\lambda = 0.01$, and calculate (37) perturbatively. Since its expression in terms of analytic functions is not very illuminating, we present, in Figure 1, the numerical results of the first two iterations, taking $\varphi(0) = 1$ and $N = 3$.

In Figure 2, we show the original zero mode (36) and the one including order-λ^2 corrections. We see that the corrected zero mode continues to be localized over the domain wall, although it is no longer a symmetric function of x_2. Besides, this perturbative method introduces only smooth corrections in the zero mode, as expected.

We conclude by summarizing that we have studied a generalization of the Callan-Harvey mechanism to the case of a non local mass in 2+1 dimensions, showing that for a certain set of assumptions about the non locality, there continues to exist a fermionic zero mode.

We have first considered a quite general non local term, deriving a linear integral equation for a function which modulates the zero mode of the local case, and accounts for the effect of the non local domain wall. Considering a defect of finite size, the Fredholm alternative theorem applies and there is a unique, localized chiral zero mode. Moreover, perturbation theory may be applied to calculate it: for the concrete example of a strongly diagonal mass, we have calculated the first few terms in a perturbative expansion, showing that they lead to negligible modifications with respect to the local mass case.

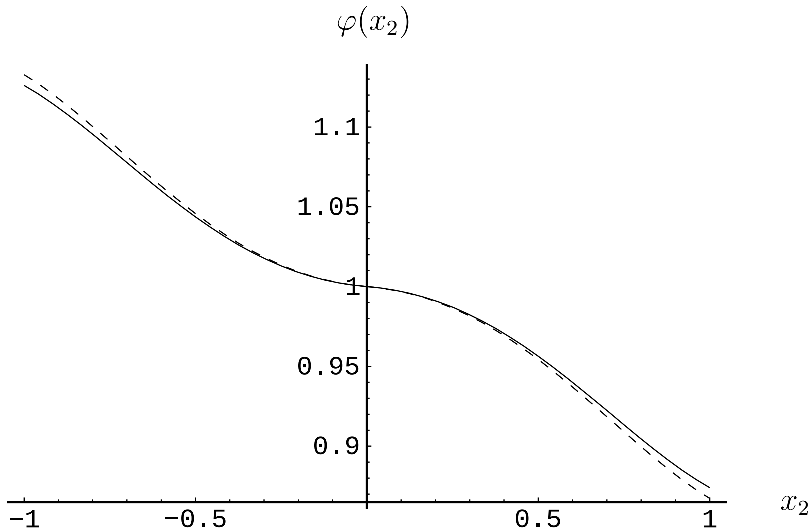

Figure 1: $\varphi(x_2)$ after two iterations of the Neumann series (33) for the x_2-compactified case. The dashed line corresponds to the first iteration.

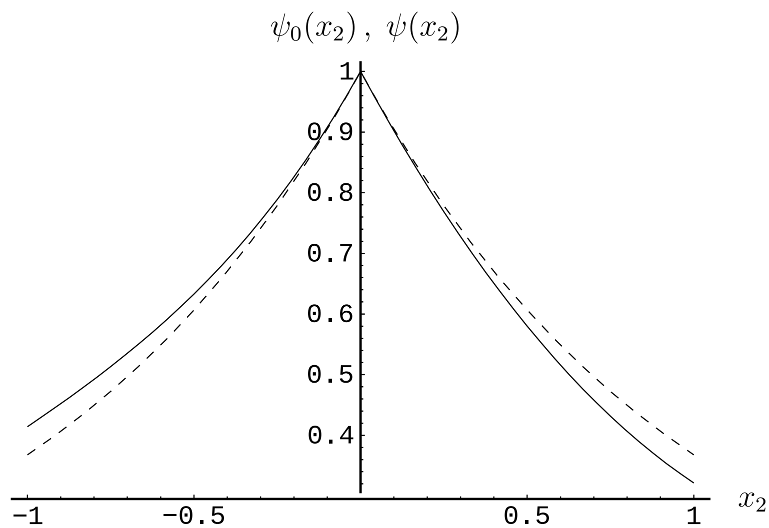

Figure 2: Zero mode profiles for local (dashed line) and non local (full line) masses.

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